SUMO - Supermodeling by combining imperfect models

Workpackage 4: Year 1

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This report describes the work done in workpackage 4 of the SUMO project in year one, i.e. task 4.1. The description of work says the following on task 4.1:

\textit{In Task 4.1 we start with the issue of the density and particular formulation of the connections:}

- \textit{How many state variables or linear combinations of the state variables (Empirical Orthogonal Functions for instance) need to be connected? We will employ the dry global quasi-geostrophic model T21QGL3 to address this issue and be guided by the work in WP1, WP2 and WP3 on this issue.}

- \textit{How should we define the connections? Most uncertainties in climate models are related to the representation of the so-called physical processes. These are the processes that determine the total diabatic heating and friction, like surface fluxes of heat, moisture and momentum, radiational fluxes, convection, latent heat release, cloud formation. Instead of connecting state variables, we can connect the models through the tendencies in the state variables that are produced by these processes. We will employ the moist quasi-geostrophic model ECBILT with relatively simple parameterisations of these physical processes to address this issue. Since this model is structurally similar to the complex models of WP5, this approach is also feasible in that case.}

- \textit{How do we need to formulate the connections to avoid disruption of the geostrophic balance and not to violate conservation laws? This issue will be addressed with the moist primitive equation model SPEEDY which is the atmospheric component of the coupled climate model SPEEDO.}

In this report we focus on the work done with the T21L3 QG model. The work on the other models was delayed because the principal investigator for WP4 (Paul Hiemstra) started 4 months later than planned in the description of work. In addition, we feel that the amount of work planned for year 1 was very ambitious. The work planned in the second year of WP4 seems to be much more realistic. Therefore, we feel confident that in year 2 we can make up for the time lost in the first year.
Summary

Connected chaotic systems can, under some circumstances, synchronize their states with limited information exchange. This is the case for toy models like the Lorenz 63. In this study we perform synchronization experiments with a quasi-geostrophic (QG) model of the atmosphere with 1449 degrees of freedom. The purpose is to determine whether connecting only a subset of the model state space can still lead to complete synchronization (CS). In addition, we evaluated whether empirical orthogonal functions (EOF) form efficient basis functions for synchronization in order to limit the number of connections. In this paper, we show that only the intermediate spectral wavenumbers (4-12) need to be connected in order to achieve CS. In addition, the minimum connection timescale needed for CS is 7.0 days. Both the connection subset and the connection timescale, or strength, can be linked to the baroclinic instabilities in the model. Using the Lorenz 63 model we show that EOFs are the optimal basis functions for synchronization. The QG model results show that the minimum number of EOFs that need to be connected for CS is a factor of three smaller than when connecting spherical harmonics. In addition, when looking at the amount of variance explained, EOFs and spherical harmonics perform equally well. This indicates that the reduced number of EOFs needed for CS is solely due to the property of EOFs of describing the variance efficiently and not due to some other dynamical property.
1 Introduction

Under certain circumstances connected chaotic systems synchronise their state (Fujisaka and Yamada, 1983; Pecora and Carroll, 1990). Synchronisation of chaos has found applications in for example the field of secure communications (Pecora et al., 1997). For the atmospheric sciences, chaos synchronisation offers interesting possibilities. For example in data assimilation, where one of the sub-systems is an atmospheric model, which is connected to another sub-system which are observations of reality (Duane et al., 2006). This form of data assimilation performs well when the models show highly non-linear behavior. In this case other linear data assimilation methods such as the Ensemble Kalman filter (Evensen, 1994) do not perform well (Simon, 2006). As a second example, long-range teleconnections can be seen as synchronisation between connected sub-systems within the larger atmospheric system. (Duane, 1997; Duane and Tribbia, 2004). A third example is synchronization multi-model approach known as super modeling (van den Berge et al., 2011). Here the ensemble of connected imperfect models more accurately follows the dynamics of the real system than each of the individual ensemble members.

An important aspect of chaos synchronisation is the amount of information exchange between the connected systems. Ideally this information exchange should be as limited as possible (Pecora et al., 1997). In this study we tried to minimize the information exchange by connecting as few state variables as possible. For the super modeling approach reducing the number of connected variables has the advantage of making the learning of the connection coefficients easier. The important question is which part of state space needs to be connected in order to achieve synchronisation. By synchronisation we mean complete synchronisation (CS). CS is achieved when the state variables of the connected systems are equal and remain equal as the systems evolve in time (Boccaletti et al., 2002).

In addition to limiting the number of connections, we also looked at limiting the connection strength. The connection strength determines how fast the models converge. This convergence has to counter the tendency of the connected systems to grow apart because of their chaotic nature. Lunkeit (2001) determined the minimum connection strength for a primitive equations model needed to attain CS. He links this connection strength to certain processes in the model that are causing the divergence between the two connected models. He did not consider limiting the number of connections between the submodels.

The standard approach is to connect the individual prognostic model variables between models. Alternatively, we propose to connect linear combinations of these model variables. Work by Yang et al. (2006) already showed the potential of this approach using bred and singular vectors. In this study we used Empirical Orthogonal Functions (EOF) (Preisendorfer and Mobley, 1988). EOFs optimize the description of variance in a given dataset by projecting onto new uncorrelated axes. Earlier studies have succesfully used EOFs as basisfunctions, to reduce the number of degrees of freedom in atmospheric models (Kwasniok, 2007; Selten, 1997). Therefore, we hypothesize that the use of EOFs will allow us a similar reduction in the number of connections needed for CS.

We defined the following research questions:
- Under what circumstances do we achieve complete synchronisation (CS)?
  - Which subset of the models’ state space needs to be connected for CS?
  - What is the minimum connecting strength needed for CS?

- Are EOFs efficient basis functions for synchronising chaotic atmospheric models?

To answer the research questions, we looked at two chaotic models: the Lorenz 63 model (Lorenz, 1963) and an intermediate complexity quasi-geostrophic (QG) atmospheric model (Marshall and Molteni, 1993). The Lorenz 63 model is a 3 variable toy model which shows chaotic behavior. It serves as an introduction to connected chaotic systems and the use of EOF analysis to efficiently connect two Lorenz models. The QG model is a much more complex and realistic model with 1449 degrees of freedom, and will serve as a more realistic experiment. The results and methods are presented separately for the Lorenz 63 model and the QG model in section 2 and 3 respectively. The study concludes with a joint discussion of the research questions in the light of our results in section 4.

2 Part A: Lorenz 63 model

2.1 Methods

2.1.1 Connected Lorenz 63 model

In order to see how our method works, we apply this method to the unidirectionally connected Lorenz 63 model (Lorenz, 1963)

\[
\begin{align*}
\dot{x}_0 &= \sigma(y_0 - x_0) \\
\dot{y}_0 &= x_0(\rho - z_0) - y_0 \\
\dot{z}_0 &= x_0y_0 - \beta z_0 \\
\dot{x}_1 &= \sigma(y_1 - x_1) + K[g \cdot (x_0 + \xi - x_1)]g_x \\
\dot{y}_1 &= x_1(\rho - z_1) - y_1 + K[g \cdot (x_0 + \xi - x_1)]g_y \\
\dot{z}_1 &= x_1y_1 - \beta z_1 + K[g \cdot (x_0 + \xi - x_1)]g_z,
\end{align*}
\]

with \(\sigma = 10\), \(\rho = 28\), and \(\beta = 8/3\). Here, \(g = (g_x, g_y, g_z)^T = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)\) with \(|g| = 1\) defines the direction of the connection and \(\xi(t)\) represents the noise in the observation which distributes uniformly in \([-\sqrt{6}, \sqrt{6}]\). This particular distribution of the noise was chosen arbitrarily. Direction of \(g\) is specified with two parameters, \(\theta\) and \(\varphi\). It is clear that the synchronizability of \(x_0\) and \(x_1\) strongly depends on the connection function \(g\) (Huang et al., 2009). Therefore, the choice of \(g(x)\) is very important.

Here, we study the synchronization error \(\langle (x_0 - x_1)^2 \rangle\) for different \((\theta, \varphi)\), and find the optimal connection function which gives the smallest synchronization error. Here, \(\langle \cdot \rangle\) denotes a time-average. We introduce the EOFs by measuring the covariance of \(x_0\), and show that the direction of the first EOF corresponding to the largest eigenvalue of the covariance matrix is close to the optimal direction of \(g\).


2.1.2 Covariance matrix and EOFs of the Lorenz 63 model

EOFs $p_i$ ($i = 1, 2, 3$) are the eigenvectors of the covariance matrix:

$$V = \begin{pmatrix}
E[(x_0 - \mu_x)(x_0 - \mu_x)] & E[(x_0 - \mu_x)(y_0 - \mu_y)] & E[(x_0 - \mu_x)(z_0 - \mu_z)] \\
E[(y_0 - \mu_y)(x_0 - \mu_x)] & E[(y_0 - \mu_y)(y_0 - \mu_y)] & E[(y_0 - \mu_y)(z_0 - \mu_z)] \\
E[(z_0 - \mu_z)(x_0 - \mu_x)] & E[(z_0 - \mu_z)(y_0 - \mu_y)] & E[(z_0 - \mu_z)(z_0 - \mu_z)]
\end{pmatrix}$$ (7)

with the eigenvalue $\lambda_i$ ($\lambda_1 > \lambda_2 > \lambda_3$). Here, $\mu_x$, $\mu_y$, and $\mu_z$ represent the time average of $x_0$, $y_0$, and $z_0$, respectively, and $E$ denotes an unbiased estimated mean value.

2.2 Results

2.2.1 EOFs for the Lorenz 63 model

We obtained the covariance matrix of $x_0$ by numerically integrating Eq. (6) as

$$V = \begin{pmatrix}
62.80 & 62.80 & 7.6 \times 10^{-4} \\
62.80 & 81.20 & -1.6 \times 10^{-2} \\
7.6 \times 10^{-4} & -1.6 \times 10^{-2} & 74.32
\end{pmatrix},$$ (8)

and EOFs as

$$p_1 = \begin{pmatrix}
0.653 \\
0.756 \\
-1.94 \times 10^{-4}
\end{pmatrix}, \quad p_2 = \begin{pmatrix}
-2.57 \times 10^{-4} \\
-3.5 \times 10^{-5} \\
-1.00
\end{pmatrix}, \quad p_3 = \begin{pmatrix}
0.756 \\
-0.653 \\
-1.71 \times 10^{-4}
\end{pmatrix},$$ (9)

as the eigenvectors of $V$ corresponding to eigenvalues $\lambda_1 = 135.47$, $\lambda_2 = 74.32$, $\lambda_3 = 8.527$, respectively. In terms of the angles $\theta$ and $\varphi$, they correspond to $p_1 = (\theta = 1.571, \varphi = 0.858)$, $p_2 = (\theta = 3.142, \varphi = 1.571)$, $p_3 = (\theta = 1.571, \varphi = 2.282)$. Note that the sign is not important for an EOF vector, the vector in the opposite direction is equally valid. So the angles given above are just one of the possible vectors. For example, for $p_1$ the second set of angles is equal to $(\theta = 1.571, \varphi = 4.000)$. Figure 1 represents the attractor of $x_0$ and the EOFs $p_i$.

2.2.2 Optimal connection function and EOF

The synchronization error in the Lorenz 63 model is plotted as a function of $\theta$ and $\varphi$ for different values of the connection strength $K$ (Fig. 2). Note that we scaled the synchronization error to have a mean of zero and a standard deviation of 2. We did this to be able to use one colorscale for all three $K$ values. As we are primarily interested in the pattern of the synchronization error, this scaling is appropriate.

For large $K$, we have minima in synchronization error at $\theta \approx 1.571, \varphi \approx 1$ and 4. These minima match the first EOF axis $p_1$, whose angles are equal to $(\theta = 1.571, \varphi = 0.858)$ and $(\theta = 1.571, \varphi = 4.000)$. Therefore, the first EOF $p_1$ gives a good approximation of the the optimal connection vector in the Lorenz 63 model.
Figure 1: The Lorenz 63 attractor (red) and its EOFs $p_1$ (green), $p_2$ (purple), and $p_3$ (blue).

Figure 2: Plot of the scaled synchronization error $\langle (x_0 - x_1)^2 \rangle$ in the $\theta$-$\varphi$ plane. The green triangles show the $\theta$ and $\varphi$ coordinates of the first EOF axis, $p_1$. 
3 Part B: Quasi-geostrophic models

3.1 Methods

3.1.1 Quasi-geostrophic model

The quasi-geostrophic (QG) model we use in this study provides a qualitatively quite realistic simulation of the atmospheric flow in Northern Hemisphere winter. The model has three levels in the vertical, with pressure as the vertical coordinate. In addition, the model uses a spectral implementation in the horizontal direction:

\[ q(\lambda, \phi, p, t) = \sum_{n=1}^{21} \sum_{m=-n}^{n} q_{mn}(p, t) Y_{mn}(\lambda, \phi) \]  

where \( \lambda \) is latitude, \( \phi \) is longitude, \( p \) is pressure (vertical coordinate), \( t \) is time, \( q_{mn} \) is the spherical harmonic coefficient and \( Y_{mn} \) are the spherical harmonic basis functions. The spherical harmonics have been truncated at total wavenumber \( n \) of 21 (T21).

The QG model is based on the conservation of quasi-geostrophic potential vorticity (Marshall and Molteni, 1993), and is described by the following equation:

\[ \frac{dq}{dt} = F(q) \]  

where the non-linear function \( F \) describes the advection of potential vorticity, the dissipation, forcing and interaction with topography. The \( q \) field is described by a sum of 483 spherical harmonic coefficients per level, leading to a total of 1449 degrees of freedom.

3.1.2 Connecting QG models

In this study we ran two QG models in parallel. The two models are connected by:

\[ \dot{q}_1 = F(q_1) + C(q_1 - q_2) \]  
\[ \dot{q}_2 = F(q_2) + C(q_2 - q_1) \]

where \( C \) is a 1449 \( \times \) 1449 matrix where the diagonal contains connection coefficients specifying the strength of the connection for each spherical harmonic. The second term on the right hand side of equation 12 is also known as the nudging term. The connection strength \( c_n \) is defined as \( 1/T_n \), where \( T_n \) is the nudging timescale. Expressing the connecting strength as a timescale allows us to compare this timescale to the timescale of processes in the model, e.g. the baroclinic instability. Comparing \( T_n \) and other timescales in the model might provide insight into why a certain nudging timescale, i.e. connecting strength, is minimally required.

In stead of connecting all state variables, we can also connect a subset of the spherical harmonics. In this study, we choose to connect a subset of the spectral space by only
connecting spherical harmonics above a certain wavenumber threshold.

### 3.1.3 Connecting using EOFs

One way of connecting subsets of state space is to only connect spherical harmonics with a certain total wave number. In this study we also looked at connecting linear combinations of spherical harmonics instead. We used Empirical Orthogonal Functions (EOF, Preisendorfer and Mobley (1988)) to select those linear combinations. EOF analysis exploits the fact that the spherical harmonics do not evolve independently, but in preferred linear combinations. The EOF patterns, or axes, capture these preferred combinations. In other fields of study, EOF analysis is also known as principal component analysis.

The EOFs are calculated by performing an eigenvalue decomposition of the covariance matrix $V$ of QG model output. Matrix $V$ is constructed by:

$$ V = \frac{1}{N-1}(\Psi - \bar{\Psi})M^2(\Psi - \bar{\Psi})^T $$  \hspace{1cm} (13)

where $N$ is the number of model output timesteps, $\Psi$ a $1449 \times N$ matrix of the streamfunction, $\bar{\Psi}$ a 1449 element vector with the sample mean for each spherical harmonic, and $M$ a $1449 \times 1449$ matrix where the diagonal equals 1 for wavenumber zero and $\sqrt{2}$ otherwise. Matrix $M$ is needed because the way the spherical harmonics are stored in the model. For wavenumbers larger than zero, there are coefficients for both positive and negative zonal numbers. In our QG model, only the coefficients of the positive zonal numbers are stored. To take into account this increased variance of the coefficients of wavenumbers larger than zero, matrix $M^2$ multiplies the appropriate coefficients by two.

The EOF analysis of a 99,000 day (369,000 time steps) QG model run resulted in the rotation matrix $E$, where the columns contain the eigenvectors of $V$. This rotation matrix can be used to project state vectors of the streamfunction onto the new EOF axes. After performing the EOF analysis, we can construct a new connection matrix $C$:

$$ C_\psi = E C_{\text{eof}} E^T M^2 $$  \hspace{1cm} (14)

where $C_{\text{eof}}$ is an $N_{\text{eof}} \times N_{\text{eof}}$ matrix where the diagonal contains the connection strength, and $N_{\text{eof}}$ the number of EOF axes used for projection. The connection matrix $C_\psi$ cannot be directly used in equation 12 because the connection is done through the potential vorticity $q$. So just before exchanging information we need to transform $q$ to $\psi$ and do the reverse operation for the resulting information exchange vector.

### 3.1.4 Experimental setup

We explored two ways of limiting which part of the model state space was connected. Firstly, by only including spherical harmonics below a certain wave number threshold ($T_{H_{\text{sph}}}$). Raising the threshold incorporates more and more fine scale structures into the connected subspace. The values for $T_{H_{\text{sph}}}$ we explored are 6-14 and 16 and 21, where 21 means that the entire state space is connected. The second way of limiting the
connection state space was by only including the EOF axes below a certain threshold ($TH_{\text{eof}}$). We explored the following values for $TH_{\text{eof}}$: 25, 39, 50, 89, 178, 220, 282, 382, 572 and 1449. Setting $TH_{\text{eof}}$ to 1449 means that the entire state space is connected. In addition to varying which part of state space was connected, we also varied the connecting timescale $T_n$, i.e. $T_n = 0.5, 1, 3, 5, 8, 16, 32$ days. Running the model for 10,000 days (40,000 timesteps) for every combination of $TH_{\text{eof}}$ or $TH_{\text{sph}}$ and $T_n$ lead to a total of 154 QG model runs.

In addition to these 154 runs, we also ran for the model for 36,000 days (144,000 timesteps) with the entire model state space connected. The aim of this runs is to explore for which $T_n$ we can still get CS, if the entire model state space is connected.

### 3.1.5 Measure of synchronisation

We describe the degree of synchronisation at time step $i$ between the two QG models as the mean distance between them in state space, i.e. the synchronization error. The synchronization error $\langle \psi_1 - \psi_2 \rangle^2$ at time step $t$ is defined as:

$$\langle \psi_1 - \psi_2 \rangle^2 = (\psi_1 - \psi_2)M^2(\psi_1 - \psi_2)$$  \hspace{1cm} (15)

where $\psi_1$ and $\psi_2$ are 1449 element streamfunction vectors with spherical harmonic coefficients for model 1 and model 2 respectively.

To quantify the degree of synchronisation in relation to a totally unsynchronised state, we propose to scale the synchronization error. We do this by dividing the synchronization error by the error between two unconnected models. The values of the scaled synchronisation error roughly fall between zero and one, where one means that the models are totally unsynchronized and 0 means CS. We estimated the unsynchronized synchronization error from a single 10,000 day QG model run. We randomly drew timesteps in the available timeseries and calculated the synchronization error between those timesteps. In this way we randomly selected points on the attractor to estimate the distribution of synchronization error in the unconnected case. The unsynchronized error was defined as the mean of this distribution. Throughout our study, when referring to the synchronization error, we mean the scaled synchronization error.

### 3.2 Results

#### 3.2.1 EOF analysis

Figure 3 shows EOF pattern index versus the cumulative explained variance. This figure shows that the explained variance grows rapidly when including more EOF patterns. Initially, 1449 variables were needed to describe the variations in streamfunction. By using EOF patterns, this number can be reduced. To explain 99% of the variability in streamfunction, we need only 784 EOF basis patterns. To explain 90%, we only need 193 EOF basis patterns. This indicates that the EOFs are a more efficient way, i.e. in less variables, of describing the streamfunction fields than using spherical harmonics.
Spatial patterns of some EOFs are shown in figure 4. The patterns are constructed by converting the columns of the rotation matrix to a gaussian grid, like is done in the QG model. The columns are 1449 element vectors containing the amplitudes of that particular EOF axis for each of the 1449 original spherical harmonics variables. We show the patterns of each layer in the model separately. The first row in figure 4 shows the first EOF axis. It shows the greatest variability around a latitude of 50 degrees, which is the location of the jet stream. The southern hemisphere (SH) shows almost no variability. This is due to the fact that our QG model was constrained to the winter period on the northern hemisphere (NH). In the NH winter, there is less variability on the SH, where it is summer. When we go to higher indicies, the patterns have smaller scales and become more noisy.
Figure 4: Basis EOF patterns for several EOFs (panel rows) for the three model layers, 200, 500 and 800 hPa (panel columns).
3.2.2 QG model experiment

Figure 5 shows the behavior of the synchronization error when increasing the subspace of state space that is connected, for a number of connection strengths $T_n$. The figure clearly shows that a much smaller amount of variables need to be connected when using EOFs in stead of spherical harmonics. This makes using EOFs a more efficient way of nudging than using spherical harmonics. In addition, when increasing $T_n$ we see that we need an increasing amount of connected variables to attain CS. From figure 5 it is clear that the minimum $T_n$ needed to get CS lies somewhere between 5 and 8. Although CS does not occur for higher $T_n$ values, the synchronization error is still smaller than one. This indicates that even a small amount of information exchange brings the submodels closer together than they would be in an unconnected system.

EOFs span the spherical harmonics space more efficiently and are independent from each other. This means that the amount of effective data exchange for $n$ number of connected spherical harmonics is lower than for the same amount of connected EOFs. A relevant question is whether EOF nudging is more efficient only because of this more efficient representation, or if EOFs are inherently more efficient. Figure 6 explores this by changing the x-axis from number of connected variables to amount of variance connected, i.e. the subset of the model state space which is connected to each other. The figure suggests that around 85% of the model state space needs to be connected in order to achieve CS. In addition, EOFs and spherical harmonics show the same development of explained variance connected versus synchronization error. This indicates that EOFs are only a more efficient expression of the spherical harmonics. So, EOFs are not inherently better at providing synchronisation.

To get a more precise estimate of the minimum connection strength required for CS we plotted $1/T_{\text{sync}}$ versus $1/T_n$. The results are shown in figure 7. We then fitted a linear regression to these points. Just before the line hits the x-axis (infinite nudging strength), there is the nudging strength for which there is CS in a finite amount of time, i.e. $T_{\text{sync}} \neq \text{Inf}$. This $T_n$ value is equal to 7.0 days.

In the previous paragraphs we compared EOFs and spherical harmonics for their synchronisation efficiency. However, in this paragraph we want to discuss which part of the model state space needs to be connected in order to achieve CS. Figure 6 shows that for a $T_n$ of 5 days, 864 spherical harmonics need to be connected. This corresponds to a wavenumber threshold of 12, with 21 being the largest wavenumber in the model. This indicates that only the large and intermediate scale structures need to be connected in order to achieve CS.

To achieve CS for $T_n$ equals 5 days using EOFs, we need to connect 572 EOFs. We mapped these EOF axes to the spherical harmonics wavenumber limit by summing the squared rows of the $1449 \times 572$ truncated rotation matrix $E$. This shows which spherical harmonics, and thus which spatial scale, is most represented in that truncated rotation matrix. The results are shown in figure 8 for the first 572 EOFs and for the remaining EOFs. The figure shows that the variance in the connected EOFs (1-572) concentrates mainly on the intermediate scale wavenumbers. The remaining variance is mainly in the small scale wavenumbers. This confirms the results for connecting spherical harmonics,
where the small scale wavenumber need not be connected. However, the EOF results also indicate that large scale wavenumbers (0-3) need not be connected.
Figure 5: Number of variables connected between the two sub-models versus the scaled synchronization error, i.e. the degree of synchronization, for several $T_n$ values for both EOF nudging and sphlimit. The shaded areas show the 95% quantile interval.

Figure 6: Explained variance versus scaled synchronization error for several $tnudge$ values for both EOF nudging and sphlimit. The shaded areas show the 95% quantile interval.
4 Discussion and conclusions

4.1 Connecting only a subset

The main purpose of this study was to determine if CS could be achieved by only connecting a subset of the model state space between the connected models. Our results clearly show that this is possible when connecting two QG models. More specifically, our results when connecting only spherical harmonics below a certain wavenumber threshold show that only the large and intermediate scale need to be connected (wavenumber smaller than 12) for CS. In addition, the large scale components (wavenumber 0-3) also need not be connected to achieve CS. The results of connecting through EOFs confirm this result. These results are in line with the work of Duane and Tribbia (2004), who also shows that low and high wavenumbers need not be connected for synchronisation.

4.2 Minimum $T_n$

The minimum value for $T_n$ to achieve CS with a fully connected system is 7.3 days. Further reduction of $T_n$ leads to a deviation from CS. The $T_n$ limit is associated with the timescale of baroclinic instabilities. If $T_n$ becomes larger, these instabilities become
more dominant and disrupt CS. Lunkeit (2001) finds a much larger $T_s$ of around 26 days. Although they use a primitive equations model and not a QG model, we have no good explanation for the fact that we have a much smaller $T_s$. However, taking into account the timescale of the baroclinic instabilities, we think our value of 7.3 days is quite realistic.

## 4.3 Efficiency of EOFs

The results in section 2 with the Lorenz model show that, in that case, EOFs are the most optimal directions for synchronisation. Our results for QG models in section 3 show that EOFs are an efficient means of synchronisation, much more efficient that just using the spherical harmonics. However, without performing a brute force analysis like we did for the Lorenz system, we cannot claim that for QG models EOFs the optimal connection strategy.

We derived the EOFs from a timeseries of model outcomes. An alternative definition of EOFs could be to fully connect two models and set the connection strength in such a way that they are almost, but not yet, in CS. Calculating EOFs of the differences in state could identify directions in state space in which the models grow apart the strongest. These could be used to connect the two models, which might prove even more efficient than our approach. This proposed new direction shares some similarities with the work on bred and singular vectors in Yang et al. (2006).
Acknowledgments

All calculation not directly related to the QG model were performed using R (R Development Core Team, 2011; Wickham, 2009, 2011).
References


